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character of the analysis one would encounter. It is curious that the ambiguities already discussed have an analogue in the following theorem:*

If P is a prime of the form $4c+1$, and $m^2+n^2=P$ (m and n integers), then either m or n has the form $\pm1(\text{mod}4)$. If $m\equiv\pm1(\text{mod}4)$, then

$$m\equiv\pm\frac{(2c)\,!}{2(c\,!)^2}(\text{mod}P).$$

Now since $(2c)\,!\equiv-1(\text{mod}P)$, we can write

$$(c\,!)^2\equiv\pm\frac{1}{2m}(\text{mod}P)\ldots\ldots(7).$$

Hence the least residue of $(c\,!)^4(\text{mod}P)$ can be determined uniquely. But in (7) the residue for $(c\,!)^2(\text{mod}P)$ is two-valued, and we face a problem which is analogous to Dirichlet's.

◆

ON THE EVALUATION OF CERTAIN DEFINITE INTEGRALS.

By PROFESSOR G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

1. In works on Calculus we find the following:

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]}dx=\tfrac{1}{2}\sqrt{\pi}\,e^{-2c}.$$

We wish to find the value of

$$\int_0^\infty e^{-[fx^4+(g/x^2)]}\sin[hx^2+(k/x^2)]dx=A,$$

$$\int_0^\infty e^{-[fx^2+(g/x^2)]}\cos[hx^2+(k/x^2)]dx=B,$$

as well as of the still more general definite integral C of §6.

In view of the substitution $y=x\sqrt{m}$, we have

$$\int_0^\infty e^{-[mx^2+(n/x^2)]}dx=\frac{1}{\sqrt{m}}\int_0^\infty e^{-[y^2+(mn/y^2)]}dy=\frac{\sqrt{\pi}}{2\sqrt{m}}e^{-2\sqrt{(mn)}}.$$

$$\text{Now } A=\frac{1}{2\sqrt{-1}}\int_0^\infty\{e^{-(f-h\sqrt{-1})x^2-(g-k\sqrt{-1})(1/x^2)}-e^{-(f+h\sqrt{-1})x^2-(g+k\sqrt{-1})(1/x^2)}\}dx.$$

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*This theorem is due to Gauss; see Smith, l. c., p. 268.

$$B=\tfrac{1}{2}\int_0^\infty[e^{-(f-h\surd-1)x^2-(g-k\surd-1)(1/x^2)}+e^{-(f+h\surd-1)x^2-(g+k\surd-1)(1/x^2)}]dx.$$

Let $\surd m=\surd[f\pm h\surd(-1)]=\surd[\tfrac{1}{2}f+\tfrac{1}{2}\surd(f^2+h^2)]\pm\surd[\tfrac{1}{2}f-\tfrac{1}{2}\surd(f^2+h^2)]=r\pm t,$

$\surd n=\surd[g\pm k\surd(-1)]=\surd[\tfrac{1}{2}g+\tfrac{1}{2}\surd(g^2+k^2)]\pm\surd[\tfrac{1}{2}g-\tfrac{1}{2}\surd(g^2+k^2)]=u\pm v.$

$$\therefore A=\frac{\surd\pi}{4\surd(-1)\ (r-t)}e^{-2(r-t)(u-v)}-\frac{\surd\pi}{4\surd(-1)\ (r+t)}e^{-2(r+t)(u+v)},$$

$$B=\frac{\surd\pi}{4(r-t)}e^{-2(r-t)(u-v)}+\frac{\surd\pi}{4(r+t)}e^{-2(r+t)(u+v)}.$$

$$A=\frac{\surd\pi}{4\surd(-1)\ (r^2-t^2)}e^{-2(ru+tv)}\ \{r[e^{2(rv+tu)}-e^{-2(rv+tu)}]+t[e^{2(rv+tu)}+e^{-2(rv+tu)}]\}$$

$$=-\frac{\surd\pi\surd-1}{2(r^2-t^2)}e^{-2(ru+tv)}\{r\sinh2(rv+tu)+t\cosh2(rv+tu)\}.$$

Similarly,

$$B=\frac{\surd\pi}{2(r^2-t^2)}e^{-2(ru+tv)}\{r\cosh2(rv+tu)+t\sinh2(rv+tu)\}.$$

$$\therefore\int_0^\infty e^{-[fx^2+(g/x^2)}\ \begin{matrix}\cos\\ \sin\end{matrix}\left[hx^2+\frac{k}{x^2}\right]dx$$

$$=\pm\frac{\surd\pi\surd(\pm1)}{2(r^2-t^2)}e^{-2(ru+tv)}\ \left[r\begin{matrix}\cosh\\ \sinh\end{matrix}\Big(2(rv+tu)\Big)+t\begin{matrix}\sinh\\ \cosh\end{matrix}\Big(2(rv+tu)\Big)\right].$$

2. Let $f=1$, $g=0$, $h=0$, $k=b^2$; then $r=1$, $t=0$, $u=\tfrac{1}{2}b\surd2$, $v=\tfrac{1}{2}b\surd-2$, $r^2-t^2=1$, $ru+tv=\tfrac{1}{2}b\surd2$, $rv+tu=\tfrac{1}{2}b\surd-2$.

$$\therefore A=-\frac{\surd\pi\surd(-1)}{2}e^{-b\surd2}\sinh(b\surd-2)=-\frac{\surd\pi}{2}e^{-b\surd2}\surd(-1)\sinh(b-\surd2)$$

$$=-\frac{\surd\pi}{2}e^{-b\surd2}\sin(b\surd-2\surd-1)=\tfrac{1}{2}\surd(\pi)e^{-b\surd2}\sin(b\surd2)=\int_0^\infty e^{-x^2}\sin(b^2/x^2)dx.$$

Similarly, $B=\int_0^\infty e^{-x^2}\cos(b^2/x^2)dx=\tfrac{1}{2}\surd(\pi)e^{-b\surd2}\cos(b\surd2).$

3. Let $f=\cos\theta$, $g=a^2\cos\theta$, $h=\sin\theta$, $k=a^2\sin\theta$; then $r=\cos\tfrac{1}{2}\theta$, $t=\surd(-1)\times\sin\tfrac{1}{2}\theta$, $u=a\cos\tfrac{1}{2}\theta$, $v=a\surd(-1)\sin\tfrac{1}{2}\theta$, $r^2-t^2=1$, $ru+tv=a\cos\theta$, $rv+tu=a\surd(-1)\sin\theta$.

$$\therefore A=\frac{\surd\pi}{2}e^{-2a\cos\theta}\{-\surd-1\ \cos\tfrac{1}{2}\theta\sinh(2a\surd-1\ \sin\theta)$$

$$-\sqrt{-1}(\sqrt{-1}\sin\tfrac{1}{2}\theta)\cosh(2a\sqrt{-1}\sin\theta)\}$$

$$=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\{\cos\tfrac{1}{2}\theta\sin(2a\sin\theta)+\sin\tfrac{1}{2}\theta\cos(2a\sin\theta)\}$$

$$=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\sin(2a\sin\theta+\tfrac{1}{2}\theta)=\int_0^\infty e^{-[x^2+(a^2/x^2)]\cos\theta}\sin\{[x^2+(a^2/x^2)]\sin\theta\}dx.$$

Similarly,

$$B=\int_0^\infty e^{-[x^2+(a^2/x^2)]\cos\theta}\cos\{[x^2+(a^2/x^2)]\sin\theta\}dx=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\cos(2a\sin\theta+\tfrac{1}{2}\theta).$$

4. Let $f=1$, $g=b^2$, $h=1$, $k=b^2$; then $r=\frac{1}{2}\sqrt{(2\sqrt{2}+2)}$, $t=\frac{1}{2}\sqrt{(2-2\sqrt{2})}$, $u=\frac{1}{2}b\sqrt{(2\sqrt{2}+2)}$, $v=\frac{1}{2}b\sqrt{(2-2\sqrt{2})}$, $r^2-t^2=\sqrt{2}$, $ru+tv=b$, $rv+tu=b\sqrt{-1}$.

$$\therefore A=\int_0^\infty e^{-[x^2+(b^2/x^2)]}\sin[x^2+(b^2/x^2)]dx$$

$$=\frac{\sqrt{\pi}}{4\sqrt{2}}e^{-2b}\{-\sqrt{-1}\sqrt{(2+2\sqrt{2})}\sinh(2b\sqrt{-1})$$
$$-\sqrt{-1}\sqrt{(2-2\sqrt{2})}\cosh(2b\sqrt{-1})\}$$

$$=\frac{\sqrt{\pi}}{4}e^{-2b}\{\sqrt{(\sqrt{2}+1)}\sin(2b)-\sqrt{(\sqrt{2}-1)}\cos(2b)\},$$

$$B=\int_0^\infty e^{-[x^2+(b^2/x^2)]}\cos[x^2+(b^2/x^2)]dx$$

$$=\frac{\sqrt{\pi}}{4}e^{-2b}\{\sqrt{(\sqrt{2}+1)}\cos(2b)-\sqrt{(\sqrt{2}-1)}\sin(2b)\}.$$

5. Let $f=\cos2\theta$, $g=\frac{1}{2}a^2\sin2\theta$, $h=\sin2\theta$, $k=\frac{1}{2}a^2\cos2\theta$; then $r=\cos\theta$, $t=\sqrt{(-1)}\sin\theta$, $u=\frac{1}{2}a\sqrt{(1+\sin2\theta)}=\frac{1}{2}a(\cos\theta+\sin\theta)$, $v=\frac{1}{2}a\sqrt{(\sin2\theta-1)}=\frac{1}{2}a\sqrt{-1}(\cos\theta-\sin\theta)$, $r^2-t^2=1$, $ru+tv=\frac{1}{2}a$, $rv+tu=\frac{1}{2}a\sqrt{-1}$.

$$\therefore A=\int_0^\infty e^{-[x^2\cos2\theta+(a^2/2x^2)\sin2\theta]}\sin[x^2\sin2\theta+(a^2/2x^2)\cos2\theta]dx$$

$$=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\{-\sqrt{(-1)}\cos\theta\sinh(a\sqrt{-1})-\sqrt{(-1)}[\sqrt{(-1)}\sin\theta]\cosh(a\sqrt{-1})\}$$

$$=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\{\cos\theta\sin a+\sin\theta\cos a)=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\sin(\theta+a),$$

$$B=\int_0^\infty e^{-[x^2\cos2\theta+(a^2/2x^2)\sin2\theta]}\cos[x^2\sin2\theta+(a^2/2x^2)\cos2\theta]dx=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\cos(\theta+a).$$

6. Let $C=\int_0^\infty e^{-(fx^2+g/x^2)}\sin(hx^2+k/x^2)\cos(px^2+q/x^2)dx$

$$=\frac{1}{4\sqrt{-1}}\int_0^\infty e^{-(fx^2+g/x^2)}\left(e^{(hx^2+k/x^2)\sqrt{-1}}-e^{-(hx^2+k/x^2)\sqrt{-1}}\right)$$

$$\times\left(e^{(px^2+q/x^2)\sqrt{-1}}+e^{-(px^2+q/x^2)\sqrt{-1}}\right)dx.$$

$$\therefore C=\frac{1}{4\sqrt{-1}}\int_0^\infty\Big[e^{-[f-(h+p)\sqrt{-1}]x^2-[g-(k-q)\sqrt{-1}]1/x^2}$$

$$-e^{-[f+(h+p)\sqrt{-1}]x^2-[g+(k+q)\sqrt{-1}]1/x^2}\Big]dx$$

$$+\frac{1}{4\sqrt{-1}}\int_0^\infty\Big[e^{-[f-(h-p)\sqrt{-1}]x^2-[g-(k-q)\sqrt{-1}]1/x^2}-e^{-[f+(h-p)\sqrt{-1}]x^2-[g+(k-q)\sqrt{-1}]1/x^2}\Big]dx.$$

Let $\sqrt{\{f\pm(h+p)\sqrt{-1}\}}=\sqrt{\{\tfrac{1}{2}f+\tfrac{1}{2}\sqrt{[f^2+(h+p)^2]}\}}$
$\pm\sqrt{\{\tfrac{1}{2}f-\tfrac{1}{2}\sqrt{[f^2+(h+p)^2]}\}}=r\pm s,$
$\sqrt{\{g\pm(k+q)\sqrt{-1}\}}=\sqrt{\{\tfrac{1}{2}g+\tfrac{1}{2}\sqrt{[g^2+(k+q)^2]}\}}$
$\pm\sqrt{\{\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{[g^2+(k+q)^2]}\}}=t\pm u,$
$\sqrt{\{f\pm(h-p)\sqrt{-1}\}}=\sqrt{\{\tfrac{1}{2}f+\tfrac{1}{2}\sqrt{[f^2+(h-p)^2]}\}}$
$\pm\sqrt{\{\tfrac{1}{2}f-\tfrac{1}{2}\sqrt{[f^2+(h-p)^2]}\}}=v\pm w,$
$\sqrt{\{g\pm(k-q)\sqrt{-1}\}}=\sqrt{\{\tfrac{1}{2}g+\tfrac{1}{2}\sqrt{[g^2+(k-q)^2]}\}}$
$\pm\sqrt{\{\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{[g^2+(k-q)^2]}\}}=y\pm z.$

$$\therefore C=\frac{\sqrt{\pi}}{8\sqrt{-1}}\left(\frac{e^{-2(r-s)(t-u)}}{r-s}-\frac{e^{-2(r+s)(t+u)}}{r+s}+\frac{e^{-2(v-w)(y-z)}}{v-w}-\frac{e^{-2(v+w)(y+z)}}{v+w}\right)$$

$$=-\frac{\sqrt{\pi}\sqrt{-1}}{4(r^2-s^2)}e^{-2(rt+su)}[r\sinh 2(ru+ts)+s\cosh 2(ru+ts)]$$

$$-\frac{\sqrt{\pi}\sqrt{-1}}{4(v^2-w^2)}e^{-2(vy+wz)}[v\sinh 2(vz+wy)+w\cosh 2(vz+wy)].$$

7. Let $f=1$, $k=a^2$, $q=b^2$, $g=h=p=0$; then $r=v=1$, $s=w=0$,

$t=\tfrac{1}{2}\sqrt{[2(a^2+b^2)]}$, $u=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2+b^2)]}$, $y=\tfrac{1}{2}\sqrt{[2(a^2-b^2)]}$,

$z=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2-b^2)]}$, $r^2-s^2=v^2-w^2=0$, $rt+su=\tfrac{1}{2}\sqrt{[2(a^2+b^2)]}$,

$ru+ts=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2+b^2)]}$, $vy+wz=\tfrac{1}{2}\sqrt{[2(a^2-b^2)]}$,

$vz+wy=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2-b^2)]}$.

$$\therefore C=\int_0^\infty e^{-x^2}\sin(a^2/x^2)\cos(b^2/x^2)dx=\frac{\sqrt{\pi}}{4}\{e^{-\sqrt{[2(a^2+b^2)]}}\sin\sqrt{[2(a^2+b^2)]}$$

$$+e^{-\sqrt{[2(a^2-b^2)]}}\sin\sqrt{[2(a^2-b^2)]}\}.$$

If $b=0$, $C=\int_0^\infty e^{-x^2}\sin(a^2/x^2)dx=\frac{\sqrt{\pi}}{2}e^{-a\sqrt{2}}\sin(a\sqrt{2})$.

8. If we integrate both sides of the equation

$$\int_0^\infty e^{-mx}dx=\frac{1}{m}$$

$(n+1)$ times with respect to m, we get

$$(-1)^{n+1}\int_0^\infty \frac{e^{-mx}}{x^{n+1}}dx=\frac{m^n}{n!}\log m-\frac{m^n}{n!}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\dots\dots+1/n).$$

From $\int_0^\infty\int_b^a e^{-mx}dxdm=\int_b^a\frac{dm}{m}$, we get $\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}dx=\log\left(\frac{b}{a}\right)$ ....(p).

From $-\int_0^\infty\int_{a+b}^{2a}\frac{e^{-mx}}{x}dxdm-\int_0^\infty\int_{a+b}^{2b}\frac{e^{-mx}}{x}dxdm=\int_{a+b}^{2a}\log m\,dm+\int_{a+b}^{2b}\log m\,dm$

we get $\int_0^\infty\left(\frac{e^{-ax}-e^{-bx}}{x}\right)^2dx=2a\log 2a+2b\log 2b-2(a+b)\log(a+b)$

$$=\log\left(\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right)\dots\dots\dots\dots(q).$$

Now $\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\cos rx\,dx=\frac{1}{2}\int_0^\infty\frac{(e^{-ax}-e^{-bx})(e^{rx\sqrt{-1}}+e^{-rx\sqrt{-2}})}{x}dx$

$$=\frac{1}{2}\int_0^\infty\left(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}+e^{-x(a+r\sqrt{-1})}-e^{-x(b+r\sqrt{-1})}\right)x^{-1}dx$$

$=\frac{1}{2}\log\left(\frac{b-r\sqrt{-1}}{a-r\sqrt{-1}}\right)+\frac{1}{2}\log\left(\frac{b+r\sqrt{-1}}{a+r\sqrt{-1}}\right)=\frac{1}{2}\log\left(\frac{b^2+r^2}{a^2+r^2}\right)$, by (p).

$$\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\sin rx\,dx=\frac{1}{2\sqrt{-1}}\int_0^\infty\left(\frac{(e^{-ax}-e^{-bx})(e^{rx\sqrt{-1}}-e^{-rx\sqrt{-1}})}{x}\right)dx$$

$$=\frac{1}{2\sqrt{-1}}\int_0^\infty\left(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}-e^{-x(a+r\sqrt{-1})}+e^{-x(b+r\sqrt{-1})}\right)x^{-1}dx$$

$$=\frac{1}{2\sqrt{-1}}\log\left(\frac{b-r\sqrt{-1}}{a-r\sqrt{-1}}\right)-\frac{1}{2\sqrt{-1}}\log\left(\frac{b+r\sqrt{-1}}{a+r\sqrt{-1}}\right)=\tan^{-1}\frac{r}{a}-\tan^{-1}\frac{r}{b}.$$

This last follows from the relations

$$\log(C\pm D\sqrt{-1})=\tfrac{1}{2}\log(C^2+D^2)\pm\sqrt{(-1)}\tan^{-1}(D/C).$$

$$\int_0^\infty\left(\frac{e^{-ax}-e^{-bx}}{x}\right)\cos^2 rx\,dx=\tfrac{1}{4}\int_0^\infty\left(\frac{(e^{-ax}-e^{-bx})\,(e^{rx\sqrt{-1}}+e^{-rx\sqrt{-1}})}{x}\right)^2 dx$$

$$=\tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}+e^{-x(a+r\sqrt{-1})}-e^{-x(b+r\sqrt{-1})})^2 dx$$

$$=\tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})})^2 dx+\tfrac{1}{2}\int_0^\infty x^{-2}(e^{-ax}-e^{-bx})^2 dx$$

$$+\tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a+r\sqrt{-1})}-e^{-x(b+r\sqrt{-1})})^2 dx$$

$$=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right]+\tfrac{1}{2}(a-r\sqrt{-1})\log[2(a-r\sqrt{-1})]$$

$$+\tfrac{1}{2}(b-r\sqrt{-1})\log[2(b-r\sqrt{-1})]-\tfrac{1}{2}(a+b-2r\sqrt{-1})\log(a+b-2r\sqrt{-1})$$

$$+\tfrac{1}{2}(a+r\sqrt{-1})\log[2(a+r\sqrt{-1})+\tfrac{1}{2}(b+r\sqrt{-1})\log[2(b+r\sqrt{-1})]$$

$$-\tfrac{1}{2}(a+b+2r\sqrt{-1})\log(a+b+2r\sqrt{-1}),\text{ from }(q),$$

$$=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right]+\tfrac{1}{2}a\log(4a^2+4r^2)+\tfrac{1}{2}b\log(4b^2+4r^2)$$

$$-\tfrac{1}{2}(a+b)\log[(a+b^2)+4r^2]$$

$$=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}(4a^2+4r^2)^a(4b^2+4r^2)^b}{(a+b)^{2(a+b)}\,[(a+b)^2+4r^2]^{(a+b)}}\right].$$

Since $\sin^2 rx=1-\cos^2 rx$, we get

$$\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^2\sin^2 rx\,dx=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}[(a+b)^2+4r^2]^{(a+b)}}{(a+b)^{2(a+b)}\,(4a^2+4r^2)^a(4b^2+4r^2)^b}\right].$$

From $\displaystyle\int_0^\infty\int_{2a+b}^{3a}\frac{e^{-mx}}{x^2}dx\,dm+2\int_0^\infty\int_{2a+b}^{a+2b}\frac{e^{-mx}}{x^2}dx\,dm+\int_0^\infty\int_{3b}^{a+2b}\frac{e^{-mx}}{x^2}dx\,dm$

$\displaystyle=\int_{2a+b}^{3a}(m\log m-m)dm+2\int_{2a+b}^{a+2b}(m\log m-m)dm+\int_{3b}^{a+2b}(m\log m-m)dm$, we get

$$\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^3 dx=-\tfrac{1}{2}(3a)^2\log(3a)+\tfrac{3}{2}(2a+b)^2\log(2a+b)$$

$$-\tfrac{3}{2}(a+2b)^2\log(a+2b)+\tfrac{1}{2}(3b)^2\log(3b)=\tfrac{1}{2}\log\left[\frac{(2a+b)^{3(2a+b)^2}(3b)^{(3b)^2}}{(3a)^{(3a)^2}(a+2b)^{3(a+2b)^2}}\right].$$

$$-\int_0^\infty\int_{3a+b}^{4a}\frac{e^{-mx}}{x^3}dx\,dm-3\int_0^\infty\int_{3a+b}^{2a+2b}\frac{e^{-mx}}{x^3}dx\,dm$$

$$-3\int_0^\infty\int_{a+3b}^{2a+2b}\frac{e^{-mx}}{x^3}dx\,dm-\int_0^\infty\int_{a+3b}^{4b}\frac{e^{-mx}}{x^3}dx\,dm$$

$$=\int_{3a+b}^{4a}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm+3\int_{3a+b}^{2a+2b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm$$

$$+3\int_{a+3b}^{2a+2b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm+\int_{a+3b}^{4b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm,$$ we get

$$\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^4dx=\tfrac{1}{6}\log\left[\frac{(4a)^{(4a)^3}(2a+2b)^{6(2a+2b)^3}(4b)^{(4b)^3}}{(3a+b)^{4(3a+b)^3}(a+3b)^{4(a+3b)^3}}\right].$$

In general, $\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^n dx$ equals

when n is even, $\frac{1}{(n-1)!}\times$

$$\log\left[\frac{[na]^{\alpha_0}[(n-2)a+2b]^{\alpha_2}\ldots\ldots[2a+(n-2)b]^{\alpha_{n-2}}[nb]^{\alpha_n}}{[(n-1)a+b]^{\alpha_2}[(n-3)a+3b]^{\alpha_3}\ldots\ldots[3a+(n-3)b]^{\alpha_{n-3}}[a+(n-1)b]^{\alpha_{n-1}}}\right];$$

and when n is odd, $\frac{1}{(n-1)!}\times$

$$\log\left[\frac{[(n-1)a+b]^{\alpha_1}[(n-3)a+3b]^{\alpha_3}\ldots\ldots[2a+(n-2)b]^{\alpha_{n-2}}[nb]^{\alpha_n}}{[na]^{\alpha_0}[(n-2)a+2b]^{\alpha_2}\ldots\ldots[3a+(n-3)]^{\alpha_{n-3}}[a+(n-1)b]^{\alpha_{n-1}}}\right],$$

where $\alpha_0=(na)^{n-1}$, $\alpha_1=n[(n-1)a+b]^{n-1}$,

$$\alpha_2=\frac{n(n-1)}{2!}[(n-2)a+2b]^{n-1},\quad \alpha_3=\frac{n(n-1)(n-2)}{3!}[(n-3)a+3b]^{n-1},$$

....

$$\alpha_i=\frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-i+1)}{i!}[(n-i)a+ib]^{n-1}.$$